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How orbits of $SU(N)$ can describe rotations in $SO(6)$

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Abstract

This paper analyses the geometry of the Lie algebra of $SO(6)$ by making use of its homomorphism with $SU(4)$. We study the vector space of 4×4 traceless, Hermitian matrices from four different viewpoints and examine the connections between them. We review the strata of this space under group transformations using established techniques for $su(N)$ algebras. We focus on orbits of special types of vectors and their interpretation as rotations of $SO(6)$ spinors.

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1. Introduction

It is now over 25 years since Michel and Radicati [1] published a remarkable paper on the geometry of the Lie algebra of $SU(3)$. It was apparently largely motivated by the importance to particle physics of the adjoint representation of $SU(3)$ in describing charges associated with $SU(3)$ currents. The work has since become an important part of the background theory for those studying various aspects of spontaneous symmetry breaking and has also received numerous citations in more abstract mathematical studies of group theory and bifurcation theory.

Despite an appendix describing how their analysis may be generalized to higher-dimensional $SU(N)$ groups, very little work has been done on applying this theory to particular $SU(N)$ groups, which may be of interest in their own right as approximate flavour symmetries (e.g. $SU(4)$ [2]) or for their role in grand unified theories (e.g. $SU(5)$ [3]). Similarly, besides the rather trivial cases of $SO(3)$ and $SO(4)$ (through its homomorphism with $SU(2)\otimes SU(2)$), the geometry of the algebras of special orthogonal groups appears to be a relatively unstudied area, at least with regard to the orbits of spinor representations. This remains true even today, when special orthogonal groups are receiving increasing attention as the isometry groups of various spaces, particularly in superstring theories [4].

This paper begins to rectify this. In the same way that the homomorphism with $SU(2)$ helps in the study of the Lie algebra of $SO(3)$ and the homomorphism with $SU(2) \otimes SU(2)$ helps in the study of the Lie algebra of $SO(4)$, the homomorphism between $SU(4)$ and $SO(6)$

allows us to apply the methods of analysing $SU(N)$ algebras presented in [1] to the Lie algebra of $SO(6)$. The group $SO(6)$ is of particular interest due to its similarity to the groups $SO(2,4)$ and $SO(1,5)$, the conformal groups in four-dimensional Minkowski and Euclidean spacetimes, and due to its role as an R-symmetry of $N = 4$ supersymmetry [5]. This paper therefore studies the vector space of all 4×4 traceless Hermitian matrices. We look at four alternative sets of basis vectors for this space: the generators of the defining representation of $SU(4)$, the generators of the two spinor representations of $SO(6)$ and the set of all matrices formed by taking products of the γ-matrices of $SO(4)$.

The rest of the paper is in four sections. The first of these is a brief summary of some well-known features of $SO(4)$, in particular the decomposition of the Lie algebra into two commuting $SU(2)$ algebras (which will be important when we come to look at $SO(6)$) and the structure obtained by taking products of γ-matrices and applying the Clifford algebra.

Section 3 reviews the method of analysis used by Michel and Radicati; it is phrased for a general $SU(N)$ group but frequently uses $SU(2)$ and $SU(3)$ as examples. We look at the action of the group on the vector space formed by linear sums of the generators and how it partitions the space into orbits. We note the existence of a symmetric and an antisymmetric algebra on this space and special orbits of 'r-vectors' and 'q-vectors'. Finally, we see how different orbits may have different stabilizers under the group action such that the orbits fall into distinct sets or strata.

In section 4 we apply this analysis to the Lie algebra of $SU(4)$. The diagonal r-vectors and q-vectors are identified and the q-vectors are shown to form an orthonormal set. The diagonal r-vectors and generators are expressed in this orthonormal basis. We remark that all the relations we are concerned with are preserved under the group action and we find the $_\vee$-relations between the q-vectors. Lastly we look at the strata [6] of the algebra and the values of the invariants for particular orbits. We do not consider the f_x- and d_x-operators—we defer this to a subsequent paper.

The final section deals with the vector space of 4×4 traceless, Hermitian matrices viewed as the Lie algebra of the rotation group $SO(6)$. This viewpoint allows the greatest insight into the geometry of this space, as we can make full use of the various rotation subalgebras. After finding the generators of the spinor representations of $SO(6)$ we are able to identify these with the 15 matrices of the Clifford algebra structure of $SO(4)$, as found in section 2, and express the diagonal q-vectors, r-vectors and generators of $SU(4)$ in terms of them. This allows a natural extension of Michel and Radicati's $_\wedge$ and $_\vee$ formalism (used to describe the antisymmetric and symmetric algebras respectively) to the Lie algebra of $SO(6)$, so we are able to study the $_\wedge$- and $_\vee$-relations between $SO(6)$ generators, making use of the geometry of the various $SO(2)$, $SO(3)$ and $SO(4)$ subalgebras. We then proceed to look more closely at the $SO(4)$ subalgebras, in particular at orbits of q-vectors and r-vectors in these subalgebras. The paper closes by examining what sets of commuting vectors can be constructed from the Lie algebra of $SO(6)$.

2. *SO*(4)

The elements of any special orthogonal group may be written

$$e^{-i\omega_{AB}T_{AB}} \tag{1}$$

where the generators T_{AB} are traceless matrices which are antisymmetric on the indices A and B, as are the parameters ω_{AB}. The set of all $\omega_{AB}T_{AB}$ form a vector space (the Lie algebra), for which the generators form a basis. If the group is unitary, the generators are also Hermitian and obey the following commutation relations:

$$[T_{AB}, T_{CD}] = -\mathrm{i}(\delta_{BC}T_{AD} - \delta_{AC}T_{BD} - \delta_{BD}T_{AC} + \delta_{AD}T_{BC}). \tag{2}$$

Rather than work with the generators directly, though, we shall deal with

$$\sigma_{AB} \equiv 2T_{AB}. \tag{3}$$

For the group $SO(4)$, the elements are then

$$\mathrm{e}^{-\mathrm{i}\omega_{\mu\nu}T_{\mu\nu}} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\sigma_{\mu\nu}} \tag{4}$$

where $\mu, \nu = 1, 2, 3, 4$ and

$$[\sigma_{\mu\nu}, \sigma_{\rho\lambda}] = -2\mathrm{i}(\delta_{\nu\rho}\sigma_{\mu\lambda} - \delta_{\mu\rho}\sigma_{\nu\lambda} - \delta_{\nu\lambda}\sigma_{\mu\rho} + \delta_{\mu\lambda}\sigma_{\nu\rho}). \tag{5}$$

2.1. Spinor representations

This group has two spinor representations, which are both two dimensional. The direct sum of these is known as the Weyl representation and for the Weyl representation it is possible to find a set of four γ-matrices which obey the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbf{1}. \tag{6}$$

With the σs given by

$$\sigma_{\mu\nu} = \frac{-\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu] \tag{7}$$

the commutation relations (5) are then automatically satisfied.

Five 4×4 matrices which obey the Clifford algebra are

$$\gamma_i = \begin{pmatrix} 0 & \mathrm{i}\sigma_i \\ -\mathrm{i}\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{8}$$

where $i = 1, 2, 3$ and σ_i are the Pauli matrices. The first four of these are taken to be the γ-matrices of $SO(4)$; thus, using (7), we have

$$\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma_{k4} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}. \tag{9}$$

The fifth is used to construct the projection operators

$$P_{\mathrm{R}} = \tfrac{1}{2}(1 + \gamma_5) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_{\mathrm{L}} = \tfrac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{10}$$

which project out the right-handed and left-handed spinors:

$$P_{\mathrm{R}}\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix} \equiv \chi_{\mathrm{R}} \qquad P_{\mathrm{R}}\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix} \equiv \sigma_{ij}^{\mathrm{R}} \tag{11}$$

$$P_{\mathrm{R}}\sigma_{k4} = \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix} \equiv \sigma_{k4}^{\mathrm{R}} \tag{12}$$

$$P_{\mathrm{L}}\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \chi_3 \\ \chi_4 \end{pmatrix} \equiv \chi_{\mathrm{L}} \qquad P_{\mathrm{L}}\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix} \equiv \sigma_{ij}^{\mathrm{L}} \tag{13}$$

$$P_{\mathrm{L}}\sigma_{k4} = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma_k \end{pmatrix} \equiv \sigma_{k4}^{\mathrm{L}}. \tag{14}$$

2.2. The homomorphism with $SU(2) \otimes SU(2)$

Now certain linear combinations of the generators generate an $SU(2)$ subgroup which acts only on the right-handed spinor:

$$\sigma_k^{\mathrm{R}} \equiv \tfrac{1}{2}\left(\tfrac{1}{2}\epsilon_{ijk}\sigma_{ij} + \sigma_{k4}\right) = \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}. \tag{15}$$

We shall denote this subgroup as $SU(2)_{\mathrm{R}}$. Similarly, the orthogonal combinations generate an $SU(2)_{\mathrm{L}}$ subgroup which acts solely on χ_{L} and therefore commutes with $SU(2)_{\mathrm{R}}$:

$$\sigma_k^{\mathrm{L}} \equiv \tfrac{1}{2}\left(\tfrac{1}{2}\epsilon_{ijk}\sigma_{ij} - \sigma_{k4}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{16}$$

(for example, $\sigma_3^{\mathrm{R}} = \frac{1}{2}(\sigma_{12} + \sigma_{34})$ and $\sigma_3^{\mathrm{L}} = \frac{1}{2}(\sigma_{12} - \sigma_{34})$).

Remembering that the generators form a basis for the vector space of all $\omega_{\mu\nu}\sigma_{\mu\nu}$, taking linear combinations in this way corresponds to changing basis in this space, from an $SO(4)$ basis to an $SU(2)_{\mathrm{R}} \otimes SU(2)_{\mathrm{L}}$ basis. We can therefore rewrite an element of the $SO(4)$ Lie algebra as an element of the $SU(2)_{\mathrm{R}} \otimes SU(2)_{\mathrm{L}}$ algebra:

$$\omega_{\mu\nu}\sigma_{\mu\nu} = \omega_{ij}(\epsilon_{ijk}\sigma_k^{\mathrm{R}} + \epsilon_{ijk}\sigma_k^{\mathrm{L}}) + 2\omega_{k4}(\sigma_k^{\mathrm{R}} - \sigma_k^{\mathrm{L}}) \tag{17}$$

$$= (\omega_{ij}\epsilon_{ijk} + 2\omega_{k4})\sigma_k^{\mathrm{R}} + (\omega_{ij}\epsilon_{ijk} - 2\omega_{k4})\sigma_k^{\mathrm{L}}. \tag{18}$$

2.3. Clifford algebra structures of $SO(4)$ and $SO(5)$

In the same way that the Pauli matrices form a basis for the space of all 2×2 traceless, Hermitian matrices, the γ-matrices of $SO(4)$ and their products form a basis for the space of all 4×4 traceless, Hermitian matrices. As this is a 15-dimensional space, we require 11 such products as well as the four γ-matrices. From the Clifford algebra, the square of any γ-matrix is just the identity, while the product of two different γs is proportional to a σ, for example:

$$\sigma_{13} = -\frac{\mathrm{i}}{2}[\gamma_1, \gamma_3] = -\mathrm{i}\gamma_1\gamma_3 \tag{19}$$

and we have seen there are six of these. Similarly, the product of all four γs is just $\pm\gamma_5$, as can be seen by multiplying them in the Weyl representation using (8). (The order in which they are multiplied can only make the difference of a sign due to the Clifford algebra.) The remaining four matrices we can get as products of three different γs, or equivalently they are products of $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$ with one of the γ_μ, e.g.

$$\mathrm{i}\gamma_2\gamma_3\gamma_4 = -\mathrm{i}\gamma_1\gamma_5 \tag{20}$$

where the factor of i ensures Hermiticity. In general, then, we define

$$\sigma_{\mu 5} \equiv \frac{-\mathrm{i}}{2}[\gamma_\mu, \gamma_5] = \frac{\mathrm{i}}{3!}\epsilon_{\mu\nu\rho\lambda}\gamma_\nu\gamma_\rho\gamma_\lambda. \tag{21}$$

The notation is deliberately suggestive: for $SO(5)$, which has one four-dimensional spinor representation, we require five γ-matrices. If we take these to be the γ_1 to γ_5 given in (8), the corresponding σs are the $\sigma_{\mu\nu}$ and $\sigma_{\mu 5}$ described above.

3. General $SU(N)$

The elements of any special unitary group may be written as

$$\mathrm{e}^{-\mathrm{i}\theta_A\lambda_A} \tag{22}$$

where the λ_A are a set of $N^2 - 1$ traceless, Hermitian $N \times N$ matrices which are twice the generators T_A. The set of all $\theta_A \lambda_A$ again form a vector space—we shall denote an arbitrary element of this vector space as x:

$$x = \theta_A \lambda_A. \tag{23}$$

On any vector space one can by definition form a scalar product of any two vectors; we follow the definition given in [1] and define the scalar product of x and y as

$$(x, y) = \tfrac{1}{2}\,\mathrm{tr}(xy). \tag{24}$$

The square of the length of the vector x is therefore

$$(x, x) = \tfrac{1}{2}\,\mathrm{tr}(x^2). \tag{25}$$

The λs form an orthonormal basis for this space:

$$(\lambda_A, \lambda_B) = \delta_{AB} \tag{26}$$

and have the product rule

$$\lambda_A \lambda_B = \frac{2}{N}\delta_{AB}\mathbf{1} + d_{ABC}\lambda_C + \mathrm{i} f_{ABC}\lambda_C \tag{27}$$

where d_{ABC} and f_{ABC} are respectively totally symmetric and totally antisymmetric under rearrangements of A, B, C.

- For example, for $SU(2)$ the λ_A are the Pauli matrices, σ_A, and $d_{ABC} = 0$ for every A, B, C, so the product rule is

$$\sigma_A \sigma_B = \delta_{AB}\mathbf{1} + \mathrm{i} f_{ABC}\sigma_C. \tag{28}$$

- For $SU(3)$, the λ_A are the Gell-Mann λs. The d_{ABC} are not all zero (this is true for all higher-dimensional $SU(N)$ as well).

We are interested in the action of the group on the Lie algebra; the action of a group element u on a vector x of the Lie algebra is given by

$$x \to x' = u x u^{-1}. \tag{29}$$

Under this action, scalar products are clearly preserved. This action is a unitary similarity transformation acting on a Hermitian matrix—it is always possible to find a u which diagonalizes x. Furthermore, if two matrices have the same eigenvalues it is possible to find a u which transforms one into the other, so all matrices with a particular set of eigenvalues lie in the same orbit under the group action.

- For $SU(2)$, all vectors of a given length, say θ, diagonalize to

$$\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} = \theta\sigma_3$$

i.e. there is just one orbit for a given length of vector.
For higher-rank $SU(N)$ this is not the case; for example in $SU(3)$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

are both unit vectors, but do not lie in the same orbit.

Two matrices have the same eigenvalues if they have the same characteristic equation. The general characteristic equation for an $N \times N$ traceless, Hermitian matrix is

$$x^N - \gamma_2(x)x^{N-2} - \gamma_3(x)x^{N-3} - \cdots - \gamma_N(x)\mathbf{1} = 0 \tag{30}$$

where the $N - 1$ invariants are defined as

$$\gamma_k(x) = \frac{1}{k}\,\mathrm{tr}\left(x^k - \sum_{l=2}^{k-2}\gamma_l(x)x^{k-l}\right) = 0. \tag{31}$$

- For $SU(2)$ this is a quadratic equation:

$$x^2 - \gamma_2(x)\mathbf{1} = 0 \tag{32}$$

with one invariant

$$\gamma_2(x) = (x, x) = \tfrac{1}{2}\,\mathrm{tr}\,x^2 \tag{33}$$

while for $SU(3)$ it is a cubic equation with the two invariants

$$\gamma_2(x) = (x, x) \qquad \text{and} \qquad \gamma_3(x) = \tfrac{1}{3}\,\mathrm{tr}\,x^3. \tag{34}$$

When the symmetric structure constants are non-zero, vectors in the space in general have non-zero anticommutators, so we can define two (linearly independent) algebras on the space, one based on the commutator:

$$x_\wedge y \equiv -\frac{\mathrm{i}}{2}\,[x, y] \tag{35}$$

and one based on the anticommutator:

$$x_\vee y \equiv \frac{\sqrt{N}}{2}\{x, y\} - \frac{1}{\sqrt{N}}\mathbf{1}\,\mathrm{tr}(xy). \tag{36}$$

This definition ensures that $x_\wedge y$ and $x_\vee y$ are both Hermitian and traceless and these relations are preserved under the group action. (The normalization is that of [1].)

3.1. r-vectors and q-vectors

For such cases ($N > 2$), there exist sets of vectors with particular values of the $N-1$ invariants which lead to a simpler characteristic equation than the general case (30). One such set is the set of unit r-vectors, defined by

$$\gamma_2(r) = 1 \qquad \gamma_3(r) = \gamma_4(r) = \cdots = 0. \tag{37}$$

For every r-vector, there is a corresponding q-vector:

$$q_r = \frac{1}{\sqrt{N-2}}r_\vee r \tag{38}$$

which has a quadratic characteristic equation:

$$q_\vee q = \frac{N-4}{\sqrt{N-2}}q. \tag{39}$$

(Michel and Radicati study these vectors in the algebra of $SU(3)$ and show that $(r, q_r) = 0$, so for any Cartan $(U(1) \otimes U(1))$ plane, a unit r-vector and its corresponding q-vector in that plane form an orthonormal basis.)

The r-vectors of any Cartan (maximal Abelian) subspace of $SU(N)$ are the roots of that space. For the diagonal Cartan subspace, which we denote as $\mathcal{C}_\mathrm{d}$, one way in which these can be found is to construct the weights using the eigenvalues of the diagonal generators and take the differences of them—we will see this for $SU(4)$ in section 4. For $SU(3)$, this procedure yields the information that one of the diagonal λs, λ_3, is a unit r-vector, with the other one, λ_8, its associated q-vector.

It is important to note that as the r-vectors are defined in terms of the invariants, under the group action an r-vector is transformed into another r-vector. Furthermore, all lengths, scalar products, $_\wedge$- and $_\vee$-relations are preserved—in particular, q-vectors are transformed into other q-vectors and an orthonormal basis is transformed into another orthonormal basis.

3.2. Orbits and strata

If a group element u commutes with a vector x, the vector is unaffected by the transformation (29). The set of all group elements u which stabilize x under this action form a group—the isotropy group or little group of x. We can always express such an element as an exponential of a second vector:

$$u = \mathrm{e}^{\mathrm{i}y} \tag{40}$$

and from considering the power expansion of this it is clear that

$$[x, u] = 0 \Leftrightarrow [x, y] = 0 \tag{41}$$

so the isotropy group of x is generated by its centralizer in the algebra.

- A diagonal q-vector of $SU(3)$, e.g.

  $$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  commutes with an $SU(2)$ group, in this case generated by

  $$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  as well as with the $U(1)$ that it generates itself. Its isotropy group is therefore $SU(2) \otimes U(1) \approx U(2)$.

  All other diagonal vectors (including, but not only, r-vectors) have all eigenvalues different, and so do not commute with an $SU(2)$—their isotropy group is therefore the Cartan subgroup $U(1) \otimes U(1)$.

Now all commutators are preserved under the group action, so if we take two vectors in the same orbit their centralizer is the same up to conjugation and therefore their isotropy group is the same up to conjugation. Indeed, the isotropy group of a vector (up to conjugation) depends only upon the multiplicities of its eigenvalues (as we saw above for the diagonal vectors of $SU(3)$), so we see the orbits partitioned into sets with equivalent isotropy groups. These sets are known as 'strata'.

(For example, we see from the above bullet point that $SU(3)$ has only two strata: one with isotropy group $SU(2) \otimes U(1)$ which contains only q-vectors and one with isotropy group $U(1) \otimes U(1)$ which contains r-vectors and other vectors with all eigenvalues different. Again this is thoroughly covered in [1].)

It is worth noting that for any $SU(N)$ there is always one stratum which has as its isotropy group $SU(N-1) \otimes U(1) \approx U(N-1)$ and one stratum which has as its isotropy group the Cartan subgroup $U(1) \otimes U(1) \ldots U(1)$ (as discussed in [6][1]).

[1] This reference briefly considers the orbits and strata of $SU(4)$, which we will be examining in detail in this paper.

4. $SU(4)$

We now want to apply all of the above theory for $SU(4)$. We can again take the λs as a basis of the 15-dimensional Lie algebra; they have the product rule

$$\lambda_A\lambda_B = \tfrac{1}{2}\delta_{AB}\mathbf{1} + d_{ABC}\lambda_C + \mathrm{i}f_{ABC}\lambda_C \tag{42}$$

so the anticommutators are non-zero:

$$\{\lambda_A, \lambda_B\} = \delta_{AB}\mathbf{1} + 2d_{ABC}\lambda_C. \tag{43}$$

For an arbitrary vector in the Lie algebra, the characteristic equation is given by

$$x^4 - \gamma_2(x)x^2 - \gamma_3(x)x - \gamma_4(x)\mathbf{1} = 0 \tag{44}$$

where

$$\gamma_2(x) = (x, x) = \tfrac{1}{2}\,\mathrm{tr}\,x^2 \qquad \gamma_3(x) = \tfrac{1}{3}\,\mathrm{tr}\,x^3 \tag{45}$$

as before and

$$\gamma_4(x) = \tfrac{1}{4}\,\mathrm{tr}(x^4 - \gamma_2(x)x^2) = \tfrac{1}{4}\,\mathrm{tr}\,x^4 - \tfrac{1}{8}(\mathrm{tr}\,x^2)^2. \tag{46}$$

As the anticommutators are non-zero, there is a $_\vee$-product given by (36) with $N = 4$:

$$x_\vee y = \{x, y\} - (x, y)\mathbf{1} \tag{47}$$

in particular,

$$x_\vee x = 2x^2 - \gamma_2(x)\mathbf{1}. \tag{48}$$

4.1. r-vectors and q-vectors of $\mathcal{C}_\mathrm{d}$

We have a set of unit r-vectors defined by (37), so their characteristic equation becomes

$$x^2(x^2 - 1) = 0. \tag{49}$$

Their eigenvalues are thus $1, -1, 0, 0$. For the diagonal Cartan subspace $\mathcal{C}_\mathrm{d}$, which in this case is three dimensional, we can show that this is in agreement with the statement that the r-vectors are the roots of the subspace by using the method outlined in section 3.1. Using the usual form of the three diagonal λs:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(for an explicit matrix representation of all fifteen based on the Gell-Mann λs of $SU(3)$, see, for example, [7]), we construct the weights of $\mathcal{C}_\mathrm{d}$:

$$\nu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \qquad \nu^2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \tag{50}$$

$$\nu^3 = \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \qquad \nu^4 = \left(0, 0, -\frac{3}{2\sqrt{6}}\right). \tag{51}$$

The roots are then just the differences of these:

$$\pm\beta^{12} = \pm\,(1, 0, 0) \qquad \pm\,\beta^{13} = \pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \tag{52}$$

$$\pm\beta^{14} = \pm\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \qquad \pm\,\beta^{23} = \pm\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \tag{53}$$

$$\pm\beta^{24} = \pm\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \qquad \pm\,\beta^{34} = \pm\left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \tag{54}$$

written as row vectors. More explicitly, the r-vectors are, for example,

$$r_1 \equiv (\beta^{23})_3\lambda_3 + (\beta^{23})_8\lambda_8 + (\beta^{23})_{15}\lambda_{15} = -\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

Similarly,

$$r_2 = \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_3 = \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{56}$$

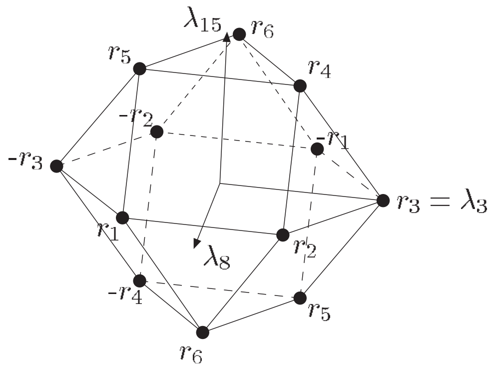
(these are the three diagonal unit r-vectors of the $SU(3)$ subgroup generated by $\lambda_1, \lambda_2, \ldots, \lambda_8$),

$$r_4 = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{57}$$

$$r_5 = -\frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{58}$$

$$r_6 = -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{59}$$

In a diagram of $\mathcal{C}_\mathrm{d}$, these roots form the familiar polyhedral root lattice:



To obtain the q-vectors of $\mathcal{C}_\mathrm{d}$, we just use (38) with $N = 4$:

$$q_1 = \frac{1}{\sqrt{2}} r_{1\vee} r_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{60}$$

$$q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{61}$$

$$q_4 = -q_1 \qquad q_5 = -q_2 \qquad q_6 = -q_3. \tag{62}$$

Note that for each of these,

$$(q_A)^2 = \tfrac{1}{2}\mathbf{1} \tag{63}$$

so each of the q_A are unit vectors, but also, from (48),

$$q_\vee q = 0 \tag{64}$$

in agreement with (39). Also, for each of the r-vectors, the associated q-vector acts as an identity (up to a factor of $1/\sqrt{2}$):

$$r_A q_A = \frac{1}{\sqrt{2}} r_A \tag{65}$$

ensuring through the tracelessness of the r-vector that r_A and q_A are orthogonal. An r-vector and its associated q-vector thus form an orthonormal basis for a plane, but to form a complete basis for $\mathcal{C}_\mathrm{d}$ we clearly need three linearly independent vectors. An obvious choice is the set q_1, q_2, q_3; by taking scalar products of these we find that they actually form an orthonormal basis. We can therefore express any vectors of the subspace as linear combinations of these three q-vectors. The vectors that we have considered so far are

$$\lambda_3 = \frac{1}{\sqrt{2}}(q_2 - q_1) \tag{66}$$

$$\lambda_8 = \frac{1}{\sqrt{6}}(2q_3 - q_1 - q_2) \tag{67}$$

$$\lambda_{15} = \frac{1}{\sqrt{3}}(q_1 + q_2 + q_3) \tag{68}$$

$$r_1 = \frac{1}{\sqrt{2}}(q_3 - q_2) \qquad r_4 = \frac{1}{\sqrt{2}}(q_3 + q_2) \tag{69}$$

$$r_2 = \frac{1}{\sqrt{2}}(q_3 - q_1) \qquad r_5 = \frac{1}{\sqrt{2}}(q_3 + q_1) \tag{70}$$

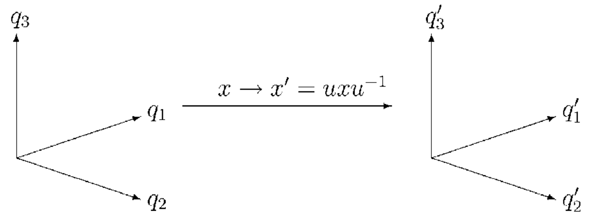
$$r_3 = \frac{1}{\sqrt{2}}(q_2 - q_1) \qquad r_6 = \frac{1}{\sqrt{2}}(q_2 + q_1). \tag{71}$$

4.2. Non-diagonal Cartan subspaces

We can consider non-diagonal Cartan subspaces by looking at what happens to $\mathcal{C}_\mathrm{d}$ under the group action. Recall that all $_\wedge$-relations are preserved under the group action, so the Cartan subalgebra is preserved. This means that $\mathcal{C}_\mathrm{d}$ is transformed into another Cartan subspace, and as stated in section 3.1, any orthonormal basis that we take for $\mathcal{C}_\mathrm{d}$ is transformed into an orthonormal basis for the new Cartan subspace.

Furthermore, as any vector x in the algebra can be diagonalized to one lying in $\mathcal{C}_\mathrm{d}$ by the action of the appropriate group element u, it follows that by applying the inverse transformation to $\mathcal{C}_\mathrm{d}$ we get the Cartan subspace containing x. Thus we can obtain any Cartan subspace by acting on $\mathcal{C}_\mathrm{d}$ with the appropriate group element.

If, as above, we take the set q_1, q_2, q_3 to be our orthonormal basis for $\mathcal{C}_\mathrm{d}$, under the group action this is transformed into another set of orthonormal q-vectors:



Hence we see that any vector can be written as a linear sum of three q-vectors:

$$q_1' = uq_1u^{-1} \tag{72}$$

$$q_2' = uq_2u^{-1} \tag{73}$$

$$q_3' = uq_3u^{-1} \tag{74}$$

with the appropriate u.

The group action also preserves the $_\vee$-relations of the vectors. As the $_\vee$-algebra is linear, we only need to consider the $_\vee$-relations of the q-vectors we are using as a basis. Using (47) and (60), (61) these are found to be

$$q_{1\vee}q_2 = -\sqrt{2}q_3 \tag{75}$$

$$q_{1\vee}q_3 = -\sqrt{2}q_2 \tag{76}$$

$$q_{2\vee}q_3 = -\sqrt{2}q_1 \tag{77}$$

or using the tensor η_{ijk} introduced in [1],

$$q_{i\vee}q_j = -\sqrt{2}\eta_{ijk}q_k. \tag{78}$$

4.3. Orbits and strata

As pointed out in [6], in $SU(4)$ there are four strata. We shall label them the q-stratum (in analogy with [1]), and the r-, s- and t-strata.

(i) q-stratum. This stratum is composed of vectors with two distinct eigenvalues, both with a multiplicity of 2. Remembering they must be traceless, this means that they must diagonalize to the form

$$d = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \tag{79}$$

which is the general form of a q-vector. Hence every vector in this stratum is a q-vector. d commutes with the $SU(2)$ group generated by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and with the $SU(2)$ group generated by

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \qquad r_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

as well as with the $U(1)$ that it generates itself. The isotropy group of the q-stratum is therefore $SU(2) \otimes SU(2) \otimes U(1)$. These results can be classified by the 4-box Young partition (2, 2).

As all q-vectors of a given length are related by similarity transformations, the q-stratum contains one orbit for each length of q-vector. An alternative way to see this is by looking at the values of the three invariants $\gamma_2(q)$, $\gamma_3(q)$ and $\gamma_4(q)$. Using the fact that each q-vector squares to $\frac{1}{2}\mathbf{1}\gamma_2(q)$ (which can be obtained from (48) and (64)), we see that

$$\gamma_3(q) = 0. \tag{80}$$

Similarly, from the characteristic equation and the square of q, we obtain

$$\gamma_4(q) = -\tfrac{1}{4}(\gamma_2(q))^2 \tag{81}$$

so two q-vectors with the same length have the same characteristic equation and therefore lie in the same orbit.

(ii) r-stratum. This stratum contains r-vectors such as

$$r_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

but it also contains other vectors with the same multiplicities of eigenvalues, such as

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

All three of these commute with the $SU(2)$ group generated by $\lambda_1, \lambda_2, \lambda_3$. They also commute with the $U(1)$ that they generate themselves as well as with a $U(1)$ group generated by one other linearly independent vector. For example, r_6 commutes with the $SU(2)$ group, its own $U(1)$ group and the $U(1)$ group generated by its associated q-vector, $q_3 = -q_6$. The isotropy group of the r-stratum is therefore $SU(2) \otimes U(1) \otimes U(1)$. These results can be classified by the 4-box Young partition $(2, 1, 1)$.

The above three matrices have different eigenvalues and therefore different characteristic equations, or equivalently different values of γ_2, γ_3 and γ_4, as can easily be verified. In particular, unit r-vectors by definition have $\gamma_3 = 0$ and $\gamma_4 = 0$. We may ask what the consequences of these conditions are (individually) for the eigenvalues. Firstly, if $\gamma_4 = 0$, the characteristic equation becomes

$$x(x^3 - \gamma_2(x)x - \gamma_3(x)) = 0 \tag{82}$$

so one eigenvalue is zero. Secondly, if $\gamma_3 = 0$, the characteristic equation becomes

$$x^4 - \gamma_2(x)x^2 - \gamma_4(x)\mathbf{1} = 0 \tag{83}$$

which is a quadratic equation in x^2, so x^2 has at most two distinct eigenvalues, both appearing twice. This implies that $x_\vee x$ is a q-vector; for the r-stratum this is only true if x is an r-vector.

Finally, if $\gamma_4 = -\gamma_3 \neq 0$, which is the case for the last of the above three matrices, the characteristic equation becomes

$$(x - 1)(x^3 + x^2 - \gamma_3(x)\mathbf{1}) = 0 \tag{84}$$

so one of the eigenvalues is 1.

(iii) s-stratum. This stratum is composed of vectors with a triple eigenvalue. From the tracelessness condition, we find that there are only eight diagonal unit vectors in this stratum:

$$s_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(-q_1 + q_2 + q_3) \tag{85}$$

$$s_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(q_1 - q_2 + q_3) \tag{86}$$

$$s_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(q_1 + q_2 - q_3) \tag{87}$$

$$s_4 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \frac{1}{\sqrt{3}}(-q_1 - q_2 - q_3) \tag{88}$$

as well as $-s_1$, $-s_2$, $-s_3$ and $-s_4 = \lambda_{15}$.

For any vector in this stratum, there is a similarity transformation which diagonalizes it to

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -3a \end{pmatrix}$$

where a is a real number. Calculating the invariants for this vector, we find they are

$$\gamma_2 = 6a^2 \qquad \gamma_3 = -8a^3 \qquad \gamma_4 = 3a^4.$$

Clearly, vectors with a triple eigenvalue a and those with a triple eigenvalue $-a$ have the same value of γ_2 (the same length) and the same value of γ_4, but their values of γ_3 have opposite signs. Thus for a given length of vector there are two distinct orbits in this stratum and γ_3 distinguishes between them. This is much the same as the situation for q-vectors in $SU(3)$, as discussed in [1].

Finally, using the same arguments as for the previous two strata, the isotropy group of this stratum is $SU(3) \otimes U(1)$ (recall that it was noted in section 3.2 that there is always such a stratum). These results can be classified by the 4-box Young partition (3, 1).

(iv) t-stratum. This is the generic stratum: it is composed of vectors with all eigenvalues different, for example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

For the second of these, $\gamma_3 = 0$. This is true for any vector of the form

$$x = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$$

so for any such vector, $x_\vee x$ is a q-vector.

Clearly vectors in this stratum only commute with the Cartan subgroup, i.e. the isotropy group is $U(1) \otimes U(1) \otimes U(1)$. These results can be classified by the 4-box Young partition (1, 1, 1, 1).

4.4. Physical examples

It is probably high time to have some physical examples and associated remarks. An appropriate place to start is the work of Girardi *et al* [8] where, although the intention was to study grand unified theories, examples directly appear which also appear in our present work. In section 5 of [8] the embedding of $SU(3) \otimes U(1)$ in $SO(6) \approx SU(4)$ is given in equation (5.4). The ultimate target in our work would be the construction of the coset space $\frac{SU(4)}{SU(3)\otimes U(1)}$ and the associated Kähler metric describing six massless Goldstone (pseudo)scalars. In the matrix $SU(4)$ form, the corresponding interpolating fields, or coordinates on the manifold, would appear in the top three entries of the final column and the first three entries of the final row, that is:

$$M = \begin{pmatrix} 0 & 0 & 0 & A^* \\ 0 & 0 & 0 & B^* \\ 0 & 0 & 0 & C^* \\ A & B & C & 0 \end{pmatrix}.$$

Note the connection between the six Goldstone Boson fields and the algebraic form of the coset space. This coset space is, of course, the well-known complex projective one in six dimensions. It is uniquely embedded in chiral $SU(4) \otimes SU(4)$ and the form of the metric is known in closed form and contains only simple functions. Since the associated generic matrix satisfies a fourth-order characteristic equation, does this mean that someone has solved the general equation of this order? No, of course not. The r- and q_r-vectors satisfy lower-order equations and the metric consequently simplifies. Now look back to section 4 of [8]. Here we see the embedding of $SU(2) \otimes SU(2) \otimes U(1)$ in $SO(6) \approx SU(4)$. This time, the Gel'fand–Zeltin [9] basis used in the previous case is generalized to orthogonal groups, and the GZ tableaux which appeared as inverted Egyptian pyramids are consequently generalized to look like inverted Mayan pyramids. This time the ultimate target in our work would be the construction of the coset space $\frac{SU(4)}{SU(2)\otimes U(2)\otimes U(1)}$ and the associated Kähler metric describing the eight massless Goldstone (pseudo)scalars. In the $SU(4)$ matrix form the interpolating fields, or general coordinates on the manifold, would appear as two blocks of four entries in the top right-hand and bottom left-hand corners:

$$M = \begin{pmatrix} 0 & 0 & A^* & C^* \\ 0 & 0 & B^* & D^* \\ A & B & 0 & 0 \\ C & D & 0 & 0 \end{pmatrix}.$$

Again note the connection between the eight Goldstone fields and the algebraic form of the coset space. Clearly, there seems to be some connection between orbit classes and classes of representations. We can display this in each separate case, but do not understand it well enough to give a description of the general case.

5. *SO*(6)

The elements of $SO(6)$ take the form (1), where A and B run $1, \ldots, 6$. With σ_{AB} defined by (3), the commutation relations for the σs look like (5) with μ, ν, ρ, λ replaced by A, B, C, D. There are two four-dimensional spinor representations; for the direct sum of these it is possible to find a set of six γ-matrices which again obey a Clifford algebra with the σs given by

$$\sigma_{AB} = \frac{-\mathrm{i}}{2}[\gamma_A, \gamma_B]. \tag{89}$$

Seven 8×8 matrices which obey the Clifford algebra are

$$\gamma_i = \begin{pmatrix} 0 & 0 & 0 & -\sigma_i \\ 0 & 0 & \sigma_i & 0 \\ 0 & \sigma_i & 0 & 0 \\ -\sigma_i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i}\mathbf{1} \\ 0 & 0 & \mathrm{i}\mathbf{1} & 0 \\ 0 & -\mathrm{i}\mathbf{1} & 0 & 0 \\ -\mathrm{i}\mathbf{1} & 0 & 0 & 0 \end{pmatrix} \tag{90}$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & \mathrm{i}\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\mathbf{1} \\ -\mathrm{i}\mathbf{1} & 0 & 0 & 0 \\ 0 & \mathrm{i}\mathbf{1} & 0 & 0 \end{pmatrix} \qquad \gamma_6 = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \tag{91}$$

$$\gamma_7 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}. \tag{92}$$

Again, we can use the first six to obtain eight-dimensional σ_{AB}s and use the last to construct the projection operators

$$P_{\mathrm{R}} = \tfrac{1}{2}(1+\gamma_7) \qquad \text{and} \qquad P_{\mathrm{L}} = \tfrac{1}{2}(1-\gamma_7) \tag{93}$$

with which we can project out the generators of the two four-dimensional spinor representations. We find that the right-handed spinor

$$\begin{pmatrix} \chi_1^{\mathrm{R}} \\ \chi_2^{\mathrm{R}} \\ \chi_3^{\mathrm{R}} \\ \chi_4^{\mathrm{R}} \end{pmatrix} = \chi^{\mathrm{R}}$$

is acted on by a representation generated by

$$\sigma_{ij}^{\mathrm{R}} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma_{i4}^{\mathrm{R}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{94}$$

$$\sigma_{i5}^{\mathrm{R}} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \sigma_{45}^{\mathrm{R}} = \begin{pmatrix} 0 & \mathrm{i}\mathbf{1} \\ -\mathrm{i}\mathbf{1} & 0 \end{pmatrix} \tag{95}$$

$$\sigma_{i6}^{\mathrm{R}} = \begin{pmatrix} 0 & \mathrm{i}\sigma_i \\ -\mathrm{i}\sigma_i & 0 \end{pmatrix} \qquad \sigma_{46}^{\mathrm{R}} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \sigma_{56}^{\mathrm{R}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{96}$$

while the generators of the left-handed spinor representation are

$$\sigma_{ij}^{\mathrm{L}} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma_{i4}^{\mathrm{L}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{97}$$

$$\sigma_{i5}^{\mathrm{L}} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \sigma_{45}^{\mathrm{L}} = \begin{pmatrix} 0 & \mathrm{i}\mathbf{1} \\ -\mathrm{i}\mathbf{1} & 0 \end{pmatrix} \tag{98}$$

$$\sigma_{i6}^{\mathrm{L}} = \begin{pmatrix} 0 & -\mathrm{i}\sigma_i \\ \mathrm{i}\sigma_i & 0 \end{pmatrix} \qquad \sigma_{46}^{\mathrm{L}} = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \qquad \sigma_{56}^{\mathrm{L}} = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{99}$$

5.1. Connections with $SO(4)$

The σ-matrices in the above two spinor representations which generate the subgroup of rotations in the first four dimensions—we will call this subgroup $\mathcal{H}$—clearly have precisely the same form as they do in the Weyl representation of $SO(4)$ (see (9)). Indeed, together with the $\sigma_{\mu 5}$ they form an $SO(5)$ subgroup, so we might expect the above $\sigma_{\mu 5}$ to be the $\sigma_{\mu 5}$ discussed in section 2.3; if we commute the appropriate γs in (8) we see that this is correct.

Now, as remarked in the introduction, the above spinor representations form (orthonormal) bases for the same space of 4×4 traceless, Hermitian matrices as the matrices of the Clifford algebra structure of $SO(4)$. Having identified some of the σs of the spinor representations with the σs of the Clifford algebra structure, the remaining σs above must be linear combinations of other matrices of the Clifford algebra structure, i.e. the γs of $SO(5)$. In fact, by inspection, we see that for the right-handed spinor, the $\sigma_{\mu 6}$ are just the γ_μ of $SO(4)$ and similarly σ_{56} is the γ_5 of $SO(4)$, with signs reversed for the left-handed spinor.

5.2. $SU(4)$ in an $SO(6)$ basis

We also know that this space is spanned by the vectors of the $SU(4)$ Lie algebra, so we should be able to couch all of the results that we obtained for $SU(4)$ in the language of $SO(6)$. It turns out that this is remarkably easy and in many ways this is the more natural description.

The obvious place to start is with the diagonal Cartan subspace. For $SU(4)$, we found three independent q-vectors in this space—these are, up to a factor, precisely the diagonal generators of $SO(6)$:

$$q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} = \frac{1}{\sqrt{2}}\sigma_{12} \tag{100}$$

$$q_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix} = -\frac{1}{\sqrt{2}}\sigma_{34} \tag{101}$$

$$q_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix} = \frac{1}{\sqrt{2}}\sigma^R_{56}. \tag{102}$$

Using (66)–(68) then gives us the transformation (change of basis) between the diagonal $SO(6)$ generators and the diagonal $SU(4)$ generators:

$$\lambda_3 = \tfrac{1}{2}(\sigma_{12} + \sigma_{34}) \tag{103}$$

$$\lambda_8 = \tfrac{1}{2\sqrt{3}}(-\sigma_{12} + \sigma_{34}) + \frac{1}{\sqrt{3}}\sigma^{\mathrm{R}}_{56} \tag{104}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}}(\sigma_{12} - \sigma_{34} + \sigma^{\mathrm{R}}_{56}). \tag{105}$$

We then have the r-vectors. r_3 and r_6 can be written in terms of q_1 and q_2, i.e. can be written in terms of the diagonal generators of $\mathcal{H}$:

$$r_3 = \frac{1}{\sqrt{2}}(q_2 - q_1) = \frac{1}{2}(\sigma_{12} + \sigma_{34}) \tag{106}$$

$$r_6 = \frac{1}{\sqrt{2}}(q_2 + q_1) = \frac{1}{2}(\sigma_{12} - \sigma_{34}). \tag{107}$$

These are what we called σ^{R}_3 and σ^{L}_3 in section 2.2. Similarly, the other diagonal r-vectors are sums and differences of the diagonal $SO(6)$ generators, for example

$$r_2 = \frac{1}{\sqrt{2}}(q_3 - q_1) = \frac{1}{2}(\sigma_{34} + \sigma^{\mathrm{R}}_{56})$$

if we consider other $SO(4)$ subgroups, e.g. in the (x_3, x_4, x_5, x_6)-space, these are then the corresponding $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ diagonal generators.

5.2.1. $_\vee$-relations between σs. Now we turn to non-diagonal vectors. A key to this is looking at $SU(2)$ subgroups of $SO(6)$. We start by noting that from the Clifford algebra, each σ-matrix of the Weyl representation squares to the identity, so for the generators of either spinor representation,

$$(\sigma_{AB})^2 = \mathbf{1}. \tag{108}$$

This has the interesting consequence that all $SO(6)$ generators are q-vectors:

$$\sigma_{AB\vee}\sigma_{AB} = 2(\sigma_{AB})^2 - \tfrac{1}{2}\,\mathrm{tr}(\sigma_{AB})^2\mathbf{1} = 2\mathbf{1} - \tfrac{1}{2}\times 4\mathbf{1} = 0. \tag{109}$$

We then observe that sets of generators such as $\{\sigma_{24}, \sigma_{25}, \sigma_{45}\}$ generate $SO(3) \approx SU(2)$ subgroups of $SO(6)$ (rotations in the (x_2, x_4, x_5)-subspace). However, we noted in section 3 that two vectors of $SU(2)$ with the same length lie in the same orbit, so all vectors in such an $SO(3)$ subgroup are related by a similarity transformation to a multiple of the generators and are therefore all q-vectors. Furthermore, it is easy to show that the $_\vee$-product of any two $SU(2)$ vectors is zero, so for any pair of generators which share an index, say σ_{AB} and σ_{AC}, their $_\vee$-product is zero as we can always find an $SO(3)$ subgroup that they fall in:

$$\sigma_{AB\vee}\sigma_{AC} = 0 \qquad \text{(no sum)}. \tag{110}$$

Now consider two generators with all indices different, for example σ_{14} and σ_{35}. They are two mutually commuting generators of an $SO(4)$ subgroup, $\langle\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{34}, \sigma_{35}, \sigma_{45}\rangle$. They satisfy $\sigma_{AB\vee}\sigma_{AB} = 0$ and $\sigma_{AB\wedge}\sigma_{CD} = 0$, so a unitary transformation can be used to transform them into another pair of mutually commuting q-vectors. We could, for example, diagonalize them to obtain σ_{12} and σ_{34}. Applying this transformation to the entire $SO(4)$ subgroup would then give us the $SO(4)$ subgroup $\mathcal{H}$ (as the commutation relations are preserved). By comparison with (75) we thus see that the $_\vee$-product of σ_{14} and σ_{35} gives us the generator which generates the $SO(2)$ subgroup orthogonal to $\langle\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{34}, \sigma_{35}, \sigma_{45}\rangle$, i.e. $\sigma_{14\vee}\sigma_{35} \propto \sigma_{26}$.

To find the proportionality, one can consider the Weyl representation. It can be shown using the Clifford algebra and the orthonormality of the σs that for A, B, C, D all different,

$$\{\sigma_{AB}, \sigma_{CD}\} - \tfrac{1}{2}\,\mathrm{tr}(\sigma_{AB}, \sigma_{CD})\mathbf{1} = -2\gamma_A\gamma_B\gamma_C\gamma_D. \tag{111}$$

For example,

$$\{\sigma_{12}, \sigma_{35}\} - \tfrac{1}{2}\,\mathrm{tr}(\sigma_{12}, \sigma_{35})\mathbf{1} = -2\gamma_1\gamma_2\gamma_3\gamma_5. \tag{112}$$

However, we also have

$$\gamma_7 = \frac{\mathrm{i}}{6!}\epsilon_{ABCDEF}\gamma_A\gamma_B\gamma_C\gamma_D\gamma_E\gamma_F = \mathrm{i}\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 = \begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix} \tag{113}$$

and we know from the Clifford algebra that the γs anticommute with each other and square to the identity. We can use this information to obtain any string of four γs such as (112) as a product of γ_7 and a σ-matrix:

$$-\gamma_7\sigma_{46} = -\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_4\gamma_6 = -\gamma_1\gamma_2\gamma_3\gamma_5. \tag{114}$$

Observe that every anticommutation introduces a minus sign, so in general we have

$$\{\sigma_{AB}, \sigma_{CD}\} - \tfrac{1}{2}\,\mathrm{tr}(\sigma_{AB}, \sigma_{CD})\mathbf{1} = \epsilon_{ABCDEF}\gamma_7\sigma_{EF} \tag{115}$$

$$= \epsilon_{ABCDEF}\begin{pmatrix}\sigma^{\mathrm{R}}_{EF} & 0\\ 0 & -\sigma^{\mathrm{L}}_{EF}\end{pmatrix} \tag{116}$$

for A, B, C, D all different. (The factor of 2 in the right-hand side of (112) results from the fact that $\gamma_7\sigma_{64}$ is also included in the sum.) This implies for the individual spinor representations, together with (110), that

$$\sigma^{\mathrm{R}}_{AB\vee}\sigma^{\mathrm{R}}_{CD} = \epsilon_{ABCDEF}\sigma^{\mathrm{R}}_{EF} \tag{117}$$

and

$$\sigma^{\mathrm{L}}_{AB\vee}\sigma^{\mathrm{L}}_{CD} = -\epsilon_{ABCDEF}\sigma^{\mathrm{L}}_{EF}. \tag{118}$$

(In the language of [10] this is saying that on changing from a basis where the group's parameters have a single vector index, such as the basis of the λs, to the $SO(6)$ basis where they have an antisymmetric pair of indices, we are replacing the totally symmetric structure constant d_{ijk} with the ϵ_{ABCDEF} tensor associated with the Pfaffian.)

Note that, among other things, these equations contain the information that if we apply the algebra (47) to the vectors of the Weyl representation of $SO(4)$ it does not close. Indeed, the Clifford algebra structure of $SO(4)$ can be defined as the minimal extension of this vector space such that the algebra does close.

5.2.2. r-vectors and q-vectors in $SO(4)$ subgroups. Now every $SO(4)$ subgroup is homomorphic to $SU(2)\otimes SU(2)$ and we know from $\mathcal{H}$ how to take orthogonal combinations of commuting $SO(4)$ generators (q-vectors) to get the $SU(2)_{\mathrm{R}}\otimes SU(2)_{\mathrm{L}}$ generators (r-vectors). We also know that all vectors of a given length in one of these $SU(2)$ subgroups lie in the same orbit, so by rotating r_3 we see that all vectors of the form $n_i^{\mathrm{R}}\sigma_i^{\mathrm{R}}$ (in the $SU(2)_{\mathrm{R}}$ subgroup) are r-vectors and similarly all $SU(2)_{\mathrm{L}}$ vectors are r-vectors. This is obviously true of the $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ vectors of any $SO(4)$ subgroup of $SO(6)$. Furthermore, by applying these $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ rotations to $q_2 = (\sigma_3^{\mathrm{R}}+\sigma_3^{\mathrm{L}})/\sqrt{2}$ independently, for example, we see that any vector that has 'equal parts' in $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$:

$$q = \theta(n_i^{\mathrm{R}}\sigma_i^{\mathrm{R}} + n_i^{\mathrm{L}}\sigma_i^{\mathrm{L}}) \tag{119}$$

(i.e. its $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ components have equal magnitude) is a q-vector.

Let us write this expression explicitly for a q-vector in $\mathcal{H}$ in terms of σ_{AB}s:

$$\begin{aligned} q = \frac{\theta}{2}(& n_1^{\mathrm{R}}(\sigma_{23}+\sigma_{14}) + n_2^{\mathrm{R}}(\sigma_{31}+\sigma_{24}) + n_3^{\mathrm{R}}(\sigma_{12}+\sigma_{34}) \\ & + n_1^{\mathrm{L}}(\sigma_{23}-\sigma_{14}) + n_2^{\mathrm{L}}(\sigma_{31}-\sigma_{24}) + n_3^{\mathrm{L}}(\sigma_{12}-\sigma_{34})) \end{aligned} \tag{120}$$

$$\begin{aligned} = \frac{\theta}{2}(&(n_1^{\mathrm{R}}+n_1^{\mathrm{L}})\sigma_{23} + (n_2^{\mathrm{R}}+n_2^{\mathrm{L}})\sigma_{31} + (n_3^{\mathrm{R}}+n_3^{\mathrm{L}})\sigma_{12} \\ & + (n_1^{\mathrm{R}}-n_1^{\mathrm{L}})\sigma_{14} + (n_2^{\mathrm{R}}-n_2^{\mathrm{L}})\sigma_{24} + (n_3^{\mathrm{R}}-n_3^{\mathrm{L}})\sigma_{34}). \end{aligned} \tag{121}$$

There are a number of interesting examples of such q-vectors obtained by equating components of n_i^{R} and n_i^{L}:

$n_i^{\mathrm{R}} = n_i^{\mathrm{L}} = n_i$: $q = \theta\,(n_1\sigma_{23} + n_2\sigma_{31} + n_3\sigma_{12})$
any element of $SU(2)_V$

$n_i^{\mathrm{R}} = -n_i^{\mathrm{L}} = n_i$: $q = \theta\,(n_1\sigma_{14} + n_2\sigma_{24} + n_3\sigma_{34})$
any element of the axial part of $\mathcal{H}$

$n_1^{\mathrm{R}} = n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = -n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = -n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{23} + n_2^{\mathrm{R}}\sigma_{24} + n_3^{\mathrm{R}}\sigma_{34})$
any element of $SO(3)$ in (x_2, x_3, x_4)-space

$n_1^{\mathrm{R}} = -n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{14} + n_2^{\mathrm{R}}\sigma_{31} + n_3^{\mathrm{R}}\sigma_{12})$
any element of $\mathcal{H}$/above $SO(3)$

$n_1^{\mathrm{R}} = -n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = -n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{14} + n_2^{\mathrm{R}}\sigma_{31} + n_3^{\mathrm{R}}\sigma_{34})$
any element of $SO(3)$ in (x_1, x_3, x_4)-space

$n_1^{\mathrm{R}} = n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = -n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{23} + n_2^{\mathrm{R}}\sigma_{24} + n_3^{\mathrm{R}}\sigma_{12})$
any element of $\mathcal{H}$/above $SO(3)$

$n_1^{\mathrm{R}} = -n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = -n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{14} + n_2^{\mathrm{R}}\sigma_{24} + n_3^{\mathrm{R}}\sigma_{12})$
any element of $SO(3)$ in (x_1, x_2, x_4)-space

$n_1^{\mathrm{R}} = n_1^{\mathrm{L}}, n_2^{\mathrm{R}} = n_2^{\mathrm{L}}, n_3^{\mathrm{R}} = -n_3^{\mathrm{L}}$: $q = \theta(n_1^{\mathrm{R}}\sigma_{23} + n_2^{\mathrm{R}}\sigma_{31} + n_3^{\mathrm{R}}\sigma_{34})$
any element of $\mathcal{H}$/above $SO(3)$.

Note that we may transform between q-vectors (or r-vectors) in a given subgroup of $SO(6)$ by acting with elements of that subgroup, but we can transform to q-vectors (or r-vectors) outside that subgroup by acting with an appropriate element of $SO(6)$.

5.2.3. Commuting sets of r and q-vectors. To clarify the geometry described above, it is informative to look at what the vectors that we are interested in commute with and what mutually commuting orthogonal sets of vectors we can construct. We have already found the centralizers of the various strata in section 4, but it would be helpful to review this in the language that we have used in this section.

Firstly we took as an example of a q-vector the matrix

$$d = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which is clearly just a multiple of σ_{56}. We observed that, besides the $U(1) \approx SO(2)$ group that it generates, it commutes with two $SU(2)$ groups, which we are now calling $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$; it should be noted that these two make up $\mathcal{H}$, the $SO(4)$ orthogonal to this subgroup. So the isotropy group is $SU(2) \otimes SU(2) \otimes U(1)$ or $SO(4) \otimes SO(2)$. For the r-vectors, we looked most closely at r_6. We noted that it commutes with the whole of $SU(2)_{\mathrm{R}}$, as well as with the $U(1)$ group that it generates itself, which is the diagonal part of $SU(2)_{\mathrm{L}}$. Finally, it commutes with its associated q-vector, which is, up to a factor, σ_{56}—this clearly acts as an identity for both $SU(2)$ groups, to which it is orthogonal.

We can now think of picking a vector in the algebra, choosing an orthogonal vector that it commutes with and then picking a third vector which is orthogonal to both of the first two and commutes with them. For example, we can start with a q-vector. We have already found that any orthogonal vector which commutes with it must lie in the algebra of the $SO(4)$ which commutes with the $SO(2)$ that it generates. However, we have the whole $SO(4)$ space to choose from, which contains q-vectors, r-vectors and vectors which are neither. If we pick one of the q-vectors in this $SO(4)$ as our second vector, the third must be a vector in the $SO(4)$ which commutes with it and is orthogonal to it. This uniquely defines a third q-vector (up to a change of length)—we can see this from σ_{12}, whose centralizer in $\mathcal{H}$ is the $U(1) \otimes U(1)$ generated by σ_{12} and σ_{34}. However, if we pick an r-vector as our second vector, this is a vector in a right (or left) $SU(2)$ subgroup of the $SO(4)$ and commutes with the whole of the left (or right) $SU(2)$ subgroup, so although we know that our third vector must be another r-vector, we have the whole of an $SU(2)$ subgroup to choose from.

Now start with an r-vector. Its isotropy group is $SU(2) \otimes U(1) \otimes U(1)$, where the algebra of the $SU(2)$ is composed entirely of r-vectors and the $U(1)$ orthogonal to the r-vector is generated by a single q-vector. We could take a q-vector from the $U(1)$ and an r-vector from the $SU(2)$, in which case the q-vector is uniquely defined (up to a change of length), whereas we are free to choose any r-vector from the $SU(2)$. We can also ask whether there are any r-vectors or q-vectors in the algebra of $SU(2) \otimes U(1)$ other than these. Take the example of r_3 which lies in $SU(2)_{\mathrm{R}}$. We know that

$$r_{3_\vee} r_3 = \sqrt{2} q_3 \qquad \text{and} \qquad r_{6_\vee} r_6 = -\sqrt{2} q_3.$$

We are asking whether there are any r-vectors or q-vectors which are linear sums of q_3 and an $SU(2)_{\mathrm{L}}$ vector. The above equations are preserved under $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ transformations. By applying these transformations to the r-vectors on the left-hand sides of these equations we can get any $SU(2)_{\mathrm{R}}$ or $SU(2)_{\mathrm{L}}$ vector. However, these transformations are in the stabilizer of q_3, so the right-hand sides are unaffected. Hence for any $SU(2)_{\mathrm{R}}$ or $SU(2)_{\mathrm{L}}$ r-vector,

$$r_\vee r = \pm \sqrt{2} q_3.$$

So the vectors that we are interested in are linear sums of an $SU(2)_{\mathrm{L}}$ r-vector and its *corresponding* q-vector. However, looking at, for example, r_1 and q_1 in the diagonal Cartan subspace, it is clear that there are no r q-vectors which are linear sums of an r-vector and its corresponding q-vector.

6. Summary

We have looked at four descriptions of the space of 4×4 traceless, Hermitian matrices. In the $SU(4)$ description we noted the existence of a stratum composed entirely of q-vectors with the simple $_\vee$-relation (64). There were also three other strata, including a stratum which contains, but is not entirely composed of, r-vectors; each of these vectors has a corresponding q-vector given by $q_r = r_\vee r/\sqrt{2}$ which it is orthogonal to.

We saw that for any Cartan subspace, there exist three unit q-vectors which form an orthonormal basis for that space and obey the $_\vee$-relation (78). The r-vectors for that Cartan subspace could then be written as differences of these q-vectors.

It emerged that all of this is easier to understand geometrically by comparing with the $SO(6)$ description of the vector space. All the generators of the $SO(6)$ spinor representations turned out to be q-vectors, with the $_\vee$-relations (117) and (118). The isotropy group of one of these generators was seen to be the direct product of the $U(1)$ that it generates itself with the $SO(4)$ subgroup that it commutes with. Each of these $SO(4)$ subgroups could be broken into a direct product of two $SU(2)$ subgroups; the vectors of the $SU(2)$ subgroups are r-vectors while the vectors with 'equal parts' in each $SU(2)$ are q-vectors. The isotropy group of one of the r-vectors was seen to be the direct product of the $U(1)$ that it generates itself, the other $SU(2)$ group in the $SO(4)$ and the $U(1)$ generated by its associated q-vector.

We have in this way managed to give a comprehensive picture of the orbits and strata of $SO(6)$ by employing techniques devised for $SU(N)$ Lie algebras and making use of the homomorphism between $SU(4)$ and $SO(6)$. We hope that this work will provide a useful insight to those working with various aspects of $SO(6)$ symmetry and this Lie algebra.

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